

# A new semidefinite relaxation for $\ell_1$ -constrained quadratic optimization and extensions

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**Abstract** In this paper, by improving the variable-splitting approach, we propose a new semidefinite programming (SDP) relaxation for the nonconvex quadratic optimization problem over the  $\ell_1$  unit ball (QPL1). It dominates the state-of-the-art SDP-based bound for (QPL1). As extensions, we apply the new approach to the relaxation problem of the sparse principal component analysis and the nonconvex quadratic optimization problem over the  $\ell_p$  ( $1 < p < 2$ ) unit ball and then show the dominance of the new relaxation.

**Keywords** Quadratic optimization · Semidefinite programming ·  $\ell_1$  unit ball · Sparse principal component analysis

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## 1 Introduction

We consider the quadratic optimization problem over the  $\ell_1$  unit ball

$$\begin{aligned} (\text{QPL1}(\mathbf{Q})) \quad & \max x^T Q x \\ \text{s.t.} \quad & \|x\|_1 \leq 1, \end{aligned}$$

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which is known as an  $\ell_1$ -norm trust-region subproblem in nonlinear programming [3] and  $\ell_1$  Grothendieck problem in combinatorial optimization [7, 8]. Applications of (QPL1(Q)) can be also found in compressed sensing where  $\|x\|_1$  is introduced to approximate  $\|x\|_0$ , the number of nonzero elements of  $x$ .

If  $Q$  is negative or positive semidefinite, (QPL1(Q)) is trivial to solve, see [13]. Generally, (QPL1(Q)) is NP-hard, even when the off-diagonal elements of  $Q$  are all nonnegative, see [6]. In the same paper, Hsia showed that (QPL1(Q)) admits an exact nonconvex semidefinite programming (SDP) relaxation, which was firstly proposed as an open problem by Pinar and Teboulle [13].

Very recently, different SDP relaxations for (QPL1(Q)) have been studied in [15]. The tightest one is the following doubly nonnegative (DNN) relaxation due to Bomze et al. [2]:

$$\begin{aligned} (\text{DNN}_{\text{L1}}(\tilde{Q})) \quad & \max \tilde{Q} \bullet Y \\ \text{s.t.} \quad & e^T Y e = 1, \\ & Y \geq 0, Y \succeq 0, Y \in \mathcal{S}^{2n} \end{aligned}$$

where  $e$  is the vector with all elements equal to 1,  $\mathcal{S}^{2n}$  is the set of  $2n \times 2n$  symmetric matrices,  $Y \geq 0$  means that  $Y$  is componentwise nonnegative,  $Y \succeq 0$  stands for that  $Y$  is positive semidefinite,  $A \bullet B = \text{trace}(AB^T) = \sum_{i,j=1}^n a_{ij}b_{ij}$  is the standard inner product of  $A$  and  $B$ , and

$$\tilde{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}.$$

Notice that the set of extreme points of  $\{x : \|x\|_1 \leq 1\}$  is  $\{e_1, -e_1, \dots, e_n, -e_n\}$ , where  $e_i$  is the  $i$ -th column of the identity matrix  $I$ . Define

$$A = [e_1, \dots, e_n, -e_1, \dots, -e_n] = [I \quad -I] \in \mathbb{R}^{n \times 2n}.$$

Then we have

$$\{x \in \mathbb{R}^n : \|x\|_1 \leq 1\} = \{Ay : e^T y = 1, y \geq 0, y \in \mathbb{R}^{2n}\}. \quad (1)$$

Consequently, (QPL1(Q)) can be equivalently transformed to the following standard quadratic program (QPS) [1]:

$$\begin{aligned} (\text{QPS}) \quad & \max_{y \in \mathbb{R}^{2n}} y^T \tilde{Q} y \\ \text{s.t.} \quad & e^T y = 1, y \geq 0. \end{aligned}$$

Now we can see that  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$  exactly corresponds to the well-known doubly nonnegative relaxation of (QPS) [2]. Moreover, as mentioned in [15],  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$  can be also derived by applying the lifting procedure [9] to the following homogeneous reformulation of (QPS):

$$\begin{aligned} & \max_{y \in \mathbb{R}^{2n}} y^T \tilde{Q} y \\ \text{s.t.} \quad & y^T e e^T y = 1, \\ & y_i y_j \geq 0, \quad i, j = 1, \dots, 2n. \end{aligned}$$

A natural extension of (QPL1(Q)) is

$$\begin{aligned} (\text{QPL2L1(Q)}) \quad & \max x^T Q x \\ \text{s.t.} \quad & \|x\|_2 = 1, \\ & \|x\|_1^2 \leq k. \end{aligned} \quad (2)$$

It is a relaxation of the sparse principal component analysis (SPCA) problem [10] obtained by replacing the original constraint  $\|x\|_0 \leq k$  with (2) due to the following fact:

$$\|x\|_1^2 \leq \|x\|_0 \|x\|_2^2 \leq k.$$

A well-known SDP relaxation for (QPL2L1(Q)) is due to d'Aspremont et al. [4]:

$$\begin{aligned} (\text{SDP}_X) \quad & \max Q \bullet X \\ \text{s.t.} \quad & \text{trace}(X) = 1, \\ & e^T |X| e \leq k, \\ & X \succeq 0, \quad X \in \mathcal{S}^n. \end{aligned}$$

Recently, Xia [15] extended the doubly nonnegative relaxation approach from (QPL1(Q)) to (QPL2L1(Q)) and obtained the following SDP relaxation:

$$\begin{aligned} (\text{DNN}_{\text{L2L1}}(\tilde{Q})) \quad & \max k \cdot \tilde{Q} \bullet Y \\ \text{s.t.} \quad & k \cdot \text{trace}(A^T A Y) = 1, \\ & e^T Y e = 1, \\ & Y \geq 0, \quad Y \succeq 0, \quad Y \in \mathcal{S}^{2n}. \end{aligned}$$

It was proved in [15] that  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) = v(\text{SDP}_X)$ , where  $v(\cdot)$  denote the optimal value of problem  $(\cdot)$ . Unfortunately, this equivalence result is incorrect though it is true that  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) \leq v(\text{SDP}_X)$ . A first counterexample will be given in this paper (see Example 2 below) to show it is possible  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) < v(\text{SDP}_X)$ .

The other extension of (QPL1(Q)) is

$$\begin{aligned} (\text{QPLp(Q)}) \quad & \max x^T Q x \\ \text{s.t.} \quad & \|x\|_p \leq 1, \end{aligned}$$

where  $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$  and  $1 < p < 2$ . (QPLp) is known as a special case of the  $\ell_p$  Grothendieck problem if the diagonal entries of  $Q$  vanish. According to the survey [7], there is no approximation and hardness results for the  $\ell_p$  Grothendieck problem with  $1 < p < 2$ . Though (QPLp(Q)) has an exact nonconvex SDP relaxation similar to that of (QPL1(Q)), the computational complexity of (QPLp(Q)) is still unknown [6].

Since the  $\ell_p$  unit balls ( $1 < p < 2$ ) are included in the  $\ell_2$  unit ball, a trivial bound for (QPLp(Q)) is

$$B_2(Q) := \max_{\|x\|_2 \leq 1} x^T Q x = \max \{ \lambda_{\max}(Q), 0 \}, \quad (3)$$

where  $\lambda_{\max}(Q)$  is the largest eigenvalue of  $Q$ .

As mentioned by Nesterov in the SDP Handbook [12], no practical SDP bounds of (QPLp(Q)) are in sight for  $1 < p < 2$ . Recently, Bomze [2] used the Hölder inequality

$$\|x\|_1 \leq \|x\|_p \|e\|_{\frac{p}{p-1}} = n^{\frac{p-1}{p}} \|x\|_p \quad (4)$$

to propose the following SDP bound

$$B_1(Q) := n^{\frac{2(p-1)}{p}} \cdot v(\text{DNN}_{L1}(\tilde{Q})). \quad (5)$$

In general,  $B_1(Q)$  dominates  $B_2(Q)$  when  $p$  close to 1, though lacking a proof.

In this paper, based on a new variable-splitting reformulation for the  $\ell_1$ -constrained set, we establish a new SDP relaxation for (QPL1(Q)), which is proved to dominate  $(\text{DNN}_{L1}(\tilde{Q}))$ . We use a small example to show the improvement could be strict. Then we extend the new approach to (QPL2L1(Q)) and obtain two new SDP relaxations. We cannot prove the first new SDP bound dominates  $(\text{DNN}_{L2L1}(\tilde{Q}))$ , though it was demonstrated by examples. However, under a mild assumption, the second new SDP bound dominates  $(\text{DNN}_{L2L1}(\tilde{Q}))$ . Finally, motivated by the model (QPL2L1(Q)), we establish a new SDP bound for (QPLp(Q)) and show it is in general tighter than  $\min\{B_2(Q), B_1(Q)\}$ .

The paper is organized as follows. In Section 1, we propose a new variable-splitting reformulation for the  $\ell_1$ -constrained set and then a new SDP relaxation for (QPL1(Q)). We show it improves the state-of-the-art SDP-based bound. In Section 2, we extend the new SDP approach to (QPL2L1(Q)) and study the obtained two new SDP relaxations. In Section 3, we establish a new SDP relaxation for (QPLp(Q)), which improves the existing upper bounds. Conclusions are made in Section 4.

## 2 A New SDP Relaxation for (QPL1(Q))

In this section, we establish a new SDP relaxation for (QPL1(Q)) based on a new variable-splitting reformulation for the  $\ell_1$ -constrained set.

For any  $x \in \mathbb{R}^n$ , let

$$\begin{aligned} y_i &= \max\{x_i, 0\}, \quad i = 1, \dots, n, \\ y_{n+i} &= -\min\{x_i, 0\}, \quad i = 1, \dots, n. \end{aligned}$$

Then we have

$$x_i = y_i - y_{i+n}, \quad i = 1, \dots, n, \quad (6)$$

$$|x_i| = y_i + y_{i+n}, \quad i = 1, \dots, n, \quad (7)$$

$$y_i y_{i+n} = 0, \quad i = 1, \dots, n, \quad (8)$$

$$y_i \geq 0, \quad i = 1, \dots, 2n. \quad (9)$$

Now we obtain a new variable-splitting reformulation of the  $\ell_1$ -constrained set:

$$\{x : \|x\|_1 \leq 1\} = \{Ay : e^T y \leq 1, y \geq 0, y \in \mathbb{R}^{2n}, y_i y_{i+n} = 0, i = 1, \dots, n\}.$$

It follows that

$$\begin{aligned} v(\text{QPL1}(Q)) &= \max_{y \in \mathbb{R}^{2n}} y^T \tilde{Q} y \\ &\quad \text{s.t. } e^T y \leq 1, y \geq 0, \\ &\quad y_i y_{i+n} = 0, i = 1, \dots, n, \\ &= \max_{y \in \mathbb{R}^{2n}} y^T \tilde{Q} y \\ &\quad \text{s.t. } e^T y y^T e \leq 1, \\ &\quad y_i y_{i+n} = 0, i = 1, \dots, n, \\ &\quad y_i y_j \geq 0, i, j = 1, \dots, 2n. \end{aligned}$$

Applying the lifting procedure [9], we obtain the following new doubly non-negative relaxation of (QPL1(Q))

$$\begin{aligned} (\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) \quad &\max \tilde{Q} \bullet Y \\ &\text{s.t. } e^T Y e \leq 1, \\ &Y_{i,n+i} = 0, i = 1, \dots, n, \\ &Y \geq 0, Y \succeq 0, Y \in \mathcal{S}^{2n}. \end{aligned}$$

We first compare the qualities of  $v(\text{DNN}_{\text{L1}})$  and  $v(\text{DNN}_{\text{L1}}^{\text{new}})$ .

**Theorem 1**  $v(\text{DNN}_{\text{L1}}(\tilde{Q})) \geq v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) \geq v(\text{QPL1}(Q))$ .

Proof. According to the definitions, we have  $v(\text{DNN}_{\text{L1}}(\tilde{Q})) \geq v(\text{QPL1}(Q))$  and  $v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) \geq v(\text{QPL1}(Q))$ . It is sufficient to prove the first inequality.

Since  $Y = 0_{2n \times 2n}$  is a feasible solution of  $(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ , we have

$$v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) \geq 0.$$

Suppose  $Q \preceq 0$ . Let  $Y^*$  be an optimal solution of  $(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ . Since  $Y^* \succeq 0$ , we have  $AY^*A^T \succeq 0$  and therefore

$$v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) = \tilde{Q} \bullet Y^* = \text{trace}((A^T Q A) Y^*) = \text{trace}(Q (A Y^* A^T)) \leq 0.$$

Consequently,  $v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) = 0$ . Similarly, we can show  $v(\text{DNN}_{\text{L1}}(\tilde{Q})) = 0$ .

Now we assume  $Q \not\preceq 0$ . There is a vector  $v$  such that  $\|v\|_1 \leq 1$  and  $v^T Q v > 0$ . That is,  $v(\text{QPL1}(Q)) > 0$ . It follows that  $v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) > 0$ . Let  $Y^*$  be an optimal solution of  $(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ . Then  $Y^* \neq 0_{2n \times 2n}$ . Moreover, since  $Y^* \succeq 0$ , we have  $e^T Y^* e > 0$ . We conclude that

$$e^T Y^* e = 1. \tag{10}$$

If this is not true, then  $0 < e^T Y^* e < 1$ . Define

$$\tilde{Y} = \frac{1}{e^T Y^* e} Y^*.$$

It is trivial to see that  $\tilde{Y}$  is also feasible to  $(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ . Moreover, we have

$$\tilde{Q} \bullet \tilde{Y} = \frac{1}{e^{TY^*e}} \tilde{Q} \bullet Y^* > \tilde{Q} \bullet Y^*,$$

which contradicts the fact that  $Y^*$  is a maximizer of  $(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ . According to the equality (10),  $Y^*$  is also a feasible solution of  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$ . Consequently,  $v(\text{DNN}_{\text{L1}}(\tilde{Q})) \geq v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ . The proof is complete.  $\square$

The following small example illustrates that  $v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$  could strictly improve  $v(\text{DNN}_{\text{L1}}(\tilde{Q}))$ .

*Example 1* Consider the following instance of dimension  $n = 6$

$$Q = \begin{bmatrix} -11 & -11 & -7 & -10 & -8 & -2 \\ -11 & -5 & -10 & -9 & -10 & -7 \\ -7 & -10 & -10 & -3 & -6 & -8 \\ -10 & -9 & -3 & -8 & -9 & -10 \\ -8 & -10 & -6 & -9 & -8 & -7 \\ -2 & -7 & -8 & -10 & -7 & -6 \end{bmatrix}$$

We modeled this instance by CVX 1.2 ([5]) and solved it by SEDUMI ([14]) within CVX. Then we obtained that

$$v(\text{DNN}_{\text{L1}}(\tilde{Q})) \approx 2.0487, \quad v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) \approx 2.0186.$$

Finally, we show that there are some cases for which  $(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$  has no improvement. This “negative” result is also interesting in the sense that in case we solve  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$ , we can fix  $Y_{i,n+i}$  ( $i = 1, \dots, n$ ) at zeros in advance.

**Theorem 2** Suppose  $\text{diag}(Q) \geq 0$ .  $v(\text{DNN}_{\text{L1}}(\tilde{Q})) = v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ .

Proof. Let  $Y^*$  be an optimal solution of  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$ . Suppose there is an index  $k \in \{1, \dots, n\}$  such that  $Y_{k,n+k}^* > 0$ . Let  $\delta_k = Y_{k,n+k}^*$  and define a symmetric matrix  $Z \in \mathcal{S}^{2n}$  where

$$Z_{kk} = Z_{n+k,n+k} = \delta_k, \quad Z_{k,n+k} = Z_{n+k,k} = -\delta_k$$

and all other elements are zeros. Then

$$Z \succeq 0, \quad \tilde{Q} \bullet Z = 2(Q_{kk} + Q_{n+k,n+k})\delta_k \geq 0.$$

It follows that

$$Y^* + Z \succeq 0, \quad Y^* + Z \geq 0, \quad (Y^* + Z)_{k,n+k} = 0, \quad \tilde{Q} \bullet (Y^* + Z) \geq \tilde{Q} \bullet Y^*.$$

Then,  $Y^* + Z$  is also an optimal solution of  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$ . Repeat the above procedure until we obtain an optimal solution of  $(\text{DNN}_{\text{L1}}(\tilde{Q}))$ , denoted by  $\tilde{Y}^*$ , satisfying  $\tilde{Y}_{i,n+i}^* = 0$  for  $i = 1, \dots, n$ . Notice that  $\tilde{Y}^*$  is a feasible solution of  $(\text{DNN}_{\text{L1}}^{\text{new}})$ . Therefore, we have  $v(\text{DNN}_{\text{L1}}(\tilde{Q})) \leq v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q}))$ . Combining this inequality with Theorem 1, we can complete the proof.  $\square$

### 3 New SDP Relaxations for (QPL2L1(Q))

In this section, we extend the above new reformulation approach to (QPL2L1(Q)) and obtain two new semidefinite programming relaxations.

Similar to the reformulation (6)-(9), we have

$$x_i = \sqrt{k}(y_i - y_{n+i}), \quad i = 1, \dots, n, \quad (11)$$

$$|x_i| = \sqrt{k}(y_i + y_{n+i}), \quad i = 1, \dots, n, \quad (12)$$

$$y_i y_{n+i} = 0, \quad i = 1, \dots, n, \quad (13)$$

$$y_i \geq 0, \quad i = 1, \dots, 2n. \quad (14)$$

It follows that

$$\begin{aligned} & \{x : \|x\|_2 = 1, \|x\|_1 \leq k\} \\ &= \{\sqrt{k}Ay : ky^T A^T Ay = 1, e^T y \leq 1, y \geq 0, y \in \mathbb{R}^{2n}, y_i y_{n+i} = 0, i = 1, \dots, n\}. \end{aligned}$$

Introducing  $Y = yy^T \succeq 0$ , we obtain the following new SDP relaxation for (QPL2L1(Q)):

$$\begin{aligned} (\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q})) : & \max k \cdot \tilde{Q} \bullet Y \\ \text{s.t. } & k \cdot \text{trace}(A^T AY) = 1, \\ & e^T Y e \leq 1, \\ & Y_{i,n+i} = 0, \quad i = 1, \dots, n, \\ & Y \geq 0, Y \succeq 0, Y \in \mathcal{S}^{2n}. \end{aligned}$$

According to the definition, we trivially have:

**Proposition 1**  $v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q})) \geq v(\text{QPL2L1}(Q)).$

**Proposition 2**  $\max \left\{ v(\text{DNN}_{\text{L2L1}}(\tilde{Q})), v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q})) \right\} \leq \lambda_{\max}(Q).$

Proof. Both  $(\text{DNN}_{\text{L2L1}}(\tilde{Q}))$  and  $(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q}))$  share the same relaxation:

$$\begin{aligned} (\text{R}_Y) \quad & \max k \cdot \tilde{Q} \bullet Y \\ \text{s.t. } & k \cdot \text{trace}(A^T AY) = 1, \\ & Y \succeq 0. \end{aligned}$$

Let  $X = kAY A^T$ . We have

$$\begin{aligned} & k \cdot \tilde{Q} \bullet Y = Q \bullet X, \\ & k \cdot \text{trace}(A^T AY) = \text{trace}(X), \\ & Y \succeq 0 \implies X \succeq 0. \end{aligned}$$

Therefore,  $(\text{R}_Y)$  can be further relaxed to

$$\begin{aligned} (\text{R}_X) \quad & \max Q \bullet X \\ \text{s.t. } & \text{trace}(X) = 1, \\ & X \succeq 0. \end{aligned}$$

Let  $Q = U\Sigma U^T$  be the eigenvalue decomposition of  $Q$ , where  $\Sigma = \text{Diag}(\sigma_1, \dots, \sigma_n)$  and  $U$  are column-orthogonal. Since

$$\text{trace}(X) = \text{trace}(U^T X U), \quad (15)$$

$$X \succeq 0 \implies X_{ii} \geq 0, \quad (16)$$

$$X \succeq 0 \iff U^T X U \succeq 0, \quad (17)$$

we can further relax  $(R_X)$  to the following linear programming problem:

$$\begin{aligned} (\text{LP}) \quad & \max \sum_{i=1}^n \sigma_i x_i \\ & \text{s.t.} \quad \sum_{i=1}^n x_i = 1, \\ & \quad x_i \geq 0, \quad i = 1, \dots, n. \end{aligned}$$

Now it is trivial to verify that

$$v(\text{LP}) = \max\{\sigma_1, \dots, \sigma_n\} = \lambda_{\max}(Q).$$

The proof is complete.  $\square$

**Corollary 1** Suppose  $v(\text{QPL2L1}(Q)) = \lambda_{\max}(Q)$ , then we have

$$v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) = v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q})) = v(\text{QPL2L1}(Q)).$$

We are unable to prove  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) \geq v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q}))$ , though we failed to have found an example such that  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) < v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q}))$ . Moreover, the following example shows that it is possible  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) > v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q}))$ . As a by-product, we observe  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) < v(\text{SDP}_X)$  from the example, which means that the result  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) = v(\text{SDP}_X)$  (Theorem 3.2 [15]) is incorrect. Notice that it is true that  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) \leq v(\text{SDP}_X)$ .

*Example 2* Consider the same instance of Example 1 and let  $k = 3$ . We modeled this instance by CVX 1.2 ([5]) and solved it by SEDUMI ([14]) within CVX. We obtained that

$$v(\text{SDP}_X) \approx 6.3104, \quad v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) \approx 6.0964, \quad v(\text{DNN}_{\text{L2L1}}^{\text{new}\leq}(\tilde{Q})) \approx 5.9962.$$

Thus, in order to theoretically improve  $v(\text{DNN}_{\text{L2L1}}(\tilde{Q}))$ , we consider

$$\begin{aligned} (\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q})) \quad & \max k \cdot \tilde{Q} \bullet Y \\ & \text{s.t.} \quad k \cdot \text{trace}(Y) = 1, \\ & \quad e^T Y e = 1, \\ & \quad Y_{i,n+i} = 0, \quad i = 1, \dots, n, \\ & \quad Y \geq 0, \quad Y \succeq 0, \quad Y \in \mathcal{S}^{2n}. \end{aligned}$$



It is trivial to see that

$$v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) \geq v(\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q})).$$

However,  $v(\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q}))$  may be not an upper bound of  $(\text{QPL2L1}(Q))$ , which is indicated by the following example.

*Example 3* Consider the same instance of Example 1 and let  $k = 5$ . We modeled this instance by CVX 1.2 ([5]) and solved it by SEDUMI ([14]) within CVX. We obtained that

$$v(\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q})) \approx 7.048 < v(\text{QPL2L1}(Q)) = \lambda_{\max}(Q) = 7.0857.$$

So, we have to identify when  $v(\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q}))$  is an upper bound of  $(\text{QPL2L1}(Q))$ .

**Theorem 3** *Suppose*

$$v(\text{QPL2L1}(Q)) < \lambda_{\max}(Q), \quad (18)$$

*we have*  $v(\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q})) \geq v(\text{QPL2L1}(Q))$ .

*Proof.* We first notice that the maximum eigenvalue problem

$$(E) \quad \max_{\|x\|_2=1} x^T Q x = \lambda_{\max}(Q)$$

is a homogeneous trust-region subproblem and hence has no local-non-global maximizer [11]. Therefore, suppose there is an optimal solution of  $(\text{QPL2L1}(Q))$ , denoted by  $x^*$ , satisfying  $\|x\|_1^2 < k$ , then  $x^*$  also globally solves (E), i.e.,

$$v(\text{QPL2L1}(Q)) = x^{*T} Q x^* = \lambda_{\max}(Q).$$

Consequently, the assumption (18) implies that

$$\begin{aligned} v(\text{QPL2L1}(Q)) &= \max x^T Q x \\ \text{s.t. } &\|x\|_2 = 1 \\ &\|x\|_1^2 = k. \end{aligned}$$

Taking the transformation (11)-(14) and then applying the lifting approach [9], we obtain the SDP relaxation  $(\text{DNN}_{\text{L2L1}}^{\text{new}=\}(\tilde{Q}))$ . The proof is complete.  $\square$

*Remark 1* The assumption (18) is generally not easy to verify. However, when  $Q$  has a unique maximum eigenvalue, (18) holds if and only if  $\|v\|_1 > \sqrt{k}$ , where  $v$  is the  $\ell_2$ -normalized eigenvector corresponding to the maximum eigenvalue of  $Q$ . Moreover, according to Corollary 1 and Proposition 2, the assumption (18) can be replaced by the following easy-to-check sufficient condition

$$v(\text{DNN}_{\text{L2L1}}(\tilde{Q})) < \lambda_{\max}(Q).$$

#### 4 A New SDP Relaxation for (QPLp(Q)) ( $1 < p < 2$ )

In this section, we first propose a new SDP relaxation for (QPLp(Q)) and then show it improves both  $B_2(Q)$  (3) and  $B_1(Q)$  (5).

Motivated by the Hölder inequality (4) and the model (QPL2L1(Q)), we obtain the following new relaxation for (QPLp(Q)):

$$\begin{aligned} (\text{QPL2L1}^{\leq}(\text{Q})) \quad & \max x^T Q x \\ \text{s.t.} \quad & \|x\|_2 \leq 1 \\ & \|x\|_1^2 \leq n^{\frac{2(p-1)}{p}}. \end{aligned}$$

Taking the transformation (11)-(14) and then applying the lifting approach [9], we obtain the following SDP relaxation for (QPL2L1<sup>≤</sup>(Q)), which is very similar to (DNN<sub>L2L1</sub><sup>new≤</sup>( $\tilde{Q}$ )):

$$\begin{aligned} (\text{DNN}_{\text{Lp}}(\tilde{Q})) \quad & \max n^{\frac{2(p-1)}{p}} \cdot \tilde{Q} \bullet Y \\ \text{s.t.} \quad & n^{\frac{2(p-1)}{p}} \cdot \text{trace}(Y) \leq 1 \\ & e^T Y e \leq 1, \\ & Y_{i,n+i} = 0, \quad i = 1, \dots, n, \\ & Y \succeq 0, \quad Y \succeq 0, \quad Y \in \mathcal{S}^{2n}. \end{aligned}$$

#### Theorem 4

$$\min\{B_2(Q), B_1(Q)\} \geq v(\text{DNN}_{\text{Lp}}(\tilde{Q})) \geq v(\text{QPLp}(Q)).$$

Proof. According to the definitions, the second inequality is trivial. It is sufficient to prove the first inequality. We first show  $B_2(Q) \geq v(\text{DNN}_{\text{Lp}}(\tilde{Q}))$ .

Let  $X = n^{\frac{2(p-1)}{p}} A Y A^T$ . Since

$$\begin{aligned} n^{\frac{2(p-1)}{p}} \cdot \tilde{Q} \bullet Y &= Q \bullet X, \\ n^{\frac{2(p-1)}{p}} \cdot \text{trace}(A^T A Y) &= \text{trace}(X), \\ Y \succeq 0 &\implies X \succeq 0, \end{aligned}$$

(DNN<sub>Lp</sub>( $\tilde{Q}$ )) has the following relaxation:

$$\begin{aligned} (\text{R}) \quad & \max Q \bullet X \\ \text{s.t.} \quad & \text{trace}(X) \leq 1, \\ & X \succeq 0. \end{aligned}$$

Let  $Q = U \Sigma U^T$  be the eigenvalue decomposition of  $Q$ , where  $\Sigma = \text{Diag}(\sigma_1, \dots, \sigma_n)$  and  $U$  are column-orthogonal. According to (15)-(17), we can further relax (R)

to the following linear programming problem:

$$\begin{aligned}
 (\text{LP}) \quad & \max \sum_{i=1}^n \sigma_i x_i \\
 \text{s.t.} \quad & \sum_{i=1}^n x_i \leq 1, \\
 & x_i \geq 0, \quad i = 1, \dots, n.
 \end{aligned}$$

It is not difficult to verify that

$$v(\text{LP}) = \max\{0, \sigma_1, \dots, \sigma_n\} = \max\{0, \lambda_{\max}(Q)\} = B_2(Q).$$

Now we prove  $B_1(Q) \geq v(\text{DNN}_{\text{LP}}(\tilde{Q}))$ . Notice that

$$\begin{aligned}
 n^{-\frac{2(p-1)}{p}} \cdot v(\text{DNN}_{\text{LP}}(\tilde{Q})) & \leq \max \tilde{Q} \bullet Y \\
 \text{s.t.} \quad & e^T Y e \leq 1, \\
 & Y_{i,n+i} = 0, \quad i = 1, \dots, n, \\
 & Y \geq 0, \quad Y \succeq 0, \quad Y \in \mathcal{S}^{2n} \\
 & = v(\text{DNN}_{\text{L1}}^{\text{new}}(\tilde{Q})) \\
 & \leq v(\text{DNN}_{\text{L1}}(\tilde{Q})),
 \end{aligned}$$

where the last inequality follows from Theorem 1. The proof is complete.  $\square$

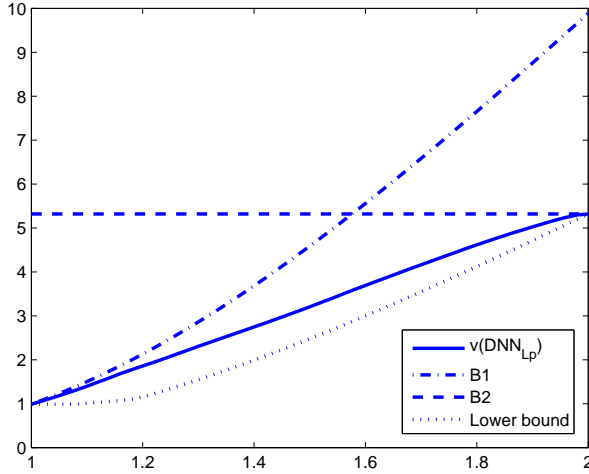
We randomly generated a symmetric matrix  $Q$  of order  $n = 10$  using the following Matlab scripts:

```
rand('state',0); Q = rand(n,n); Q = (Q+Q')/2;
```

and then compared the qualities of the three upper bounds,  $v(\text{DNN}_{\text{LP}}(\tilde{Q}))$ ,  $B_1(Q)$  and  $B_2(Q)$ . The results were plotted in Figure 1, where the lower bound of QPLp(Q) is computed as follows. Solve  $(\text{DNN}_{\text{LP}}(\tilde{Q}))$  and obtain the optimal solution  $Y^*$ . Let  $y, z$  be the unit eigenvectors corresponding to the maximum eigenvalues of  $AY^*A^T$  and  $Q$ , respectively. Then  $\frac{1}{\|y\|_p}y$  and  $\frac{1}{\|z\|_p}z$  are two feasible solutions of  $(\text{QPLp}(Q))$  and

$$\max \left\{ \frac{y^T Q y}{\|y\|_p^2}, \frac{z^T Q z}{\|z\|_p^2} \right\}$$

gives a lower bound of  $v(\text{QPLp}(Q))$ . From Figure 1, we can see that for  $1 < p < 2$ , though  $B_2(Q)$  and  $B_1(Q)$  cannot dominate each other, both are strictly improved by  $v(\text{DNN}_{\text{LP}}(\tilde{Q}))$ .



**Fig. 1** Quality of the lower bound and the three upper bounds,  $B_2(Q)$ ,  $B_1(Q)$  and  $v(\text{DNN}_{L_p}(\tilde{Q}))$  in dependence of  $p$ .

## 5 Conclusion

The SDP relaxation has been known to generate high quality bounds for nonconvex quadratic optimization problems. In this paper, based on a new variable-splitting characterization of the  $\ell_1$  unit ball, we establish a new semidefinite programming (SDP) relaxation for the quadratic optimization problem over the  $\ell_1$  unit ball (QPL1). We show the new developed SDP bound dominates the state-of-the-art SDP-based upper bound for (QPL1). There is an example to show the improvement could be strict. Then we extend the new reformulation approach to the relaxation problem of the sparse principal component analysis (QPL2L1) and obtain two SDP formulations. Examples demonstrate that the first SDP bound is in general tighter than the DNN relaxation for (QPL2L1). But we are unable to prove it. Under a mild assumption, the second SDP bound dominates the DNN relaxation. Finally, we extend our approach to the nonconvex quadratic optimization problem over the  $\ell_p$  ( $1 < p < 2$ ) unit ball (QPLp) and show the new SDP bound dominates two upper bounds in recent literature.

## References

1. I.M. Bomze, M. Dür, E. De Klerk, C. Roos, A.J. Quist, T. Terlaky, On copositive programming and standard quadratic optimization problems, *Journal of Global Optimization*, 18: 301–320, 2000
2. I.M. Bomze, F. Frommlet, M. Rubey, Improved SDP bounds for minimizing quadratic functions over the  $\ell^1$ -ball, *Optimization Letters*, 1(1): 49–59, 2007

3. A.R. Conn, N.I.M. Gould, P.L. Toint. Trust-Region Methods. MPS/SIAM Series on Optimization. SIAM, Philadelphia, PA, 2000
4. A. d'Aspremont, L. El Ghaoui, M.I. Jordan, G. R. G. Lanckriet, A direct formulation for sparse PCA using semidefinite programming, *SIAM Review*, 48(3):434–448, 2007
5. M. Grant, S. Boyd (2010) CVX: Matlab software for disciplined convex programming, version 1. 21 Web. <http://cvxr.com/cvx>
6. Y. Hsia, Complexity and Nonlinear Semidefinite Programming Reformulation of  $\ell_1$ -constrained Nonconvex Quadratic Optimization, *Optimization Letters*, DOI 10.1007/s11590-013-0670-1, 2014
7. S. Khot, A. Naor, Grothendieck-type inequalities in combinatorial optimization, *Communications on Pure and Applied Mathematics*, 65(7), 992–1035, 2012
8. Kindler, G., Naor, A., Schechtman, G.: The UGC hardness threshold of the Grothendieck problem, *Math. Oper. Res.* 35(2), 267–283, 2010
9. L. Lovasz, A. Schrijver, Cones of matrices and set-functions and 0-1 optimization, *SIAM J. Optimization* 1: 166–190, 1991
10. R. Luss, M. Teboulle, Convex Approximations to Sparse PCA via Lagrangian Duality, *Operations Research Letters*, 39(1): 57–61, 2011
11. J. M. Martínez, Local minimizers of quadratic functions on Euclidean balls and spheres, *SIAM J. Optim.* 4: 159–176, 1994
12. Y. Nesterov, Global Quadratic Optimization via Conic Relaxation, in *Handbook of Semidefinite Programming*, H. Wolkowicz, R. Saigal and L. Vandenberghe, eds., Kluwer Academic Publishers, Boston, 363–387, 2000
13. M.Ç. Pinar, M. Teboulle, On semidefinite bounds for maximization of a non-convex quadratic objective over the  $\ell_1$  unit ball, *RAIRO-Operations Research*, 40(3): 253–265, 2006
14. J.F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, *Optimization Methods and Software*, 11–12: 625–653, 1999
15. Y. Xia, New results on semidefinite bounds for  $\ell_1$ -constrained nonconvex quadratic optimization, *RAIRO-Operations Research*, 47(3): 285–297, 2013

